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ON A METHOD OF SYNTHESIS OF NETWORKS

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ON A METHOD OF SYNTHESIS OF NETWORKS[a]

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A description of a certain general method for the synthesis of networks – for example, contact networks, contact parallel-series networks, and networks of functional elements – is given, and the previously known estimates of the complexity of these networks are improved upon.

AUTHOR

One of the problems of cybernetics [translator's note: U. S. usage = electronics] is the problem of constructing networks which realize prescribed functions out of comparatively simple elements. This might be, for example, the synthesis of electronic networks out of standard blocks, or relay contact networks out of standard relays, etc. In doing this, one generally attempts to construct, in one sense or another, the *optimum* networks.

In many cases, there are trivial procedures for finding such extremal networks; however, they are not effective enough in the sense that they involve very extensive computations, and, moreover, give no advance indication of the complexity of the network which will be obtained. Consequently, the question arises of discovering a more effective method for constructing sufficiently good networks with estimates [bounds] on the complexity.

A function for estimating the complexity of networks, evaluating objects from some sequence $\mathfrak{N} = \mathfrak{N}_1, \mathfrak{N}_2, \ldots, \mathfrak{N}_n, \ldots$ of collections of objects (in our case, $\mathfrak{N}_n$ is the collection of all algebraic logic functions of η arguments), can be introduced in the following manner. Let each network $\mathcal{S}$ under consideration be placed in correspondence with the real number $L(\mathcal{S})$ – the "index of simplicity" (requiring that the index of

[a] Sections 2 and 3 have not been translated here.

simplicity characterize the complexity of the network). We examine the following function: $L(f) = \inf L(\mathcal{S})$ (where *inf* is taken over all networks $\mathcal{S}$ which realize the function f); and $L(n) = \sup_{f \in \mathfrak{N}_n} L(f)$.[*1]

The problem of network synthesis with estimates has been investigated by numerous authors [1, 5, 7, 8, 9, 10, 12, 14, 16, 20 *et al*]. Here it is intended to direct attention to one universal method[*2] for three cases: in § 2, for the synthesis of contact networks (the previously known upper estimate [20] is reduced twofold); in § 3, for the synthesis of contract Π -networks (the previously known upper estimate is substantially lowered); in § 6, for the synthesis of networks out of functional elements, this problem for one particular case is resolved with some significant finality[*3] (the definition of such networks is given in §4).

[*1] The function $L(\eta)$ was introduced by C. E. Shannon [20] for estimating the number of contacts in a contact network.

[*2] This method was previously applied by the author to the synthesis of diode and contact-diode networks [3].

[*3] Here an asymptotic formula for $L(\eta)$ is obtained, whereas in the work of D. O. Muller [16] there are upper and lower bounds for this function. For details concerning formulation of the problem, see § 4.

1. THE "PROPER REPRESENTATION" OF THE FUNCTIONS OF ALGEBRAIC LOGIC

Every function of algebraic logic f can be exhibited by a table with binary entries (*cf.* Table 1).

Table 1

				0 ⋮ 0	$\sigma_{i_{k+1}}$ ⋮ σ_{i_n}	1 ⋮ 1	$x_{i_{k+1}}$ ⋮ x_{i_n}
x_{i_1}	$\cdots$	$x_{i_{k-1}}$	x_{i_k}				
0	$\cdots$	0	0			A_1	
						A_2	
						A_1	
σ_{i_1}	$\cdots$	$\sigma_{i_{k=1}}$	σ_{i_k}	$\cdots$ $f(\sigma_1, \cdots, \sigma_n)$			
						A_p	
1	$\cdots$	1	1			A_r	

In this paragraph, the number k and the arguments $x_{i_1}, \cdots, x_{i_k}; x_{i_{k+1}}, \cdots, x_{i_n}$ listed in the Table will be regarded as fixed. The matrix defining the meaning of the function f will be denoted by $M(f)$.

We dissociate the rows of the matrix $M(f)$ into groups $A_1, \cdots, A_p$ (*cf.* Table 1). We denote by $f_j(x_1, \cdots, x_n)$ the function which coincides with $f(x_1, \cdots, x_n)$ for the rows of group A_j in $M(f)$ and equals 0 in the remaining cases.[*4] Thus,

$$f(x_1, \cdots, x_n) = \bigvee_{j=1}^{n} f_j(x_1, \cdots, x_n).$$

[*4] We will say in this case that the function f_j is restricted to the arguments $x_{i_1}, \cdots, x_{i_k}$.

Let us consider the matrix $M(f_j)$. Its columns are dissociated into groups equally among its own columns [let the number of groups be equal to $t(j)$]: we will enumerate these groups. The function f_j can be represented in the form

$$f_j = \bigvee_{k=1}^{t(j)} f_{jh},$$

where f_{jh} coincides with f_j in the kth group of columns and equals 0 in the remaining cases. It is apparent that

$$f_{jh}(x_1, \cdots, x_n) = f_{jh}^{(1)}(x_{i_1}, \cdots, x_{i_k}) \cdot f_{jh}^{(2)}(x_{i_{k+1}}, \cdots, x_{i_n}),$$

where[*5]

$$f_{jh}^{(1)}(x_{i_1}, \cdots, x_{i_k}) = \bigvee x_{i_1}^{\sigma_{i_1}} \cdots x_{i_k}^{\sigma_{i_k}}$$

[regarded disjunctively on the collection $\Sigma_{jh}^{(1)}$ of sets $(\sigma_{i_1}, \cdots, \sigma_{i_k})$, which correspond to the non-zero rows of the matrix $M(f_{jh})$];

$$f_{jh}^{(2)}(x_{i_{k+1}}, \cdots, x_{i_n}) = \bigvee x_{i_{k+1}}^{\sigma_{i_{k+1}}} \cdots x_{i_n}^{\sigma_{i_n}}$$

[regarded disjunctively on the collection $\Sigma_{jh}^{(2)}$ of sets $(\sigma_{i_{k+1}}, \cdots, \sigma_{i_n})$, which correspond to the non-zero columns of the matrix $M(f_{jh})$].

Thus,

$$f(x_1, \cdots, x_n) \bigvee_{j=1}^{p} f_j = \bigvee_{j=1}^{p} \bigvee_{k=1}^{t(j)} \left[\left(\bigvee x_{i_1}^{\sigma_1} \cdots x_{i_k}^{\sigma_k}\right) \cdot \left(\bigvee x_{i_{k+1}}^{\sigma_{i_{k+1}}} \cdots x_{i_n}^{i_n}\right)\right] \tag{1.1}$$

$$(\sigma_{i_1}, \cdots, \sigma_{i_k}) \in \Sigma_{jh}^{1} \qquad (\sigma_{i_{k+1}}, \cdots, \sigma_{i_n}) \in \Sigma_{jh}^{2}.$$

[*5] x^σ will denote x if $\sigma = 1$, and $\bar{x}$ if $\sigma = 0$.

The representation (1.1) will be called a *proper* representation of the function f, if the groups $A_1, \cdots, A_p$ have an equal number s of rows (although they could be, for one thing, contained in a smaller number of rows). The numbers k and s will be termed the *parameters* of this representation.

4. NETWORKS OF FUNCTIONAL ELEMENTS

We will examine the networks [2] constructed from functional elements*6, that is, elementary subnetworks [2], having any number of input terminals and one output terminal; every function is realized out of elementary subnetworks from the arguments, the collection of which equals the collection of input terminals of these subnetworks (Fig. 6)*7. We will give a definition of the notions of *networks, vertices*, and *input and output terminal networks*.

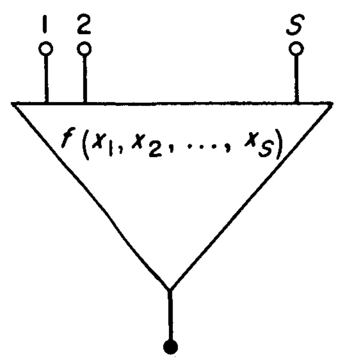

Fig. 6

DEFINITION*8 (by induction)

1. Functional elements will be networks; the corresponding input-terminal and output-terminal functional elements appear at their input terminals and output terminals; the collection of their vertices coincides with the collection of all the terminals.

2. If S' and S'' are networks without common vertices, their union is a network; the collection of its vertices, input, and output terminals, is derived by joining (set-theoretically) the corresponding collections of networks S' and S'' (*cf.* Fig. 7a).

3. If S is a network, the result of identifying ("fusing") some inputs of a terminal is likewise a network (S'). The collection of vertices, input, and output terminals of the network S'

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*6 In some cases we will drop the word "functional."

*7 The number of input terminals of the functional elements can be equal to zero; in that case, one realizes functions of 0 arguments, *i.e.*, constants.

*8 This definition appears for the special case of defining networks, given in [2]. With modifications, it approximates the definition of *logical sets*, introduced by Burks and Wright [13].

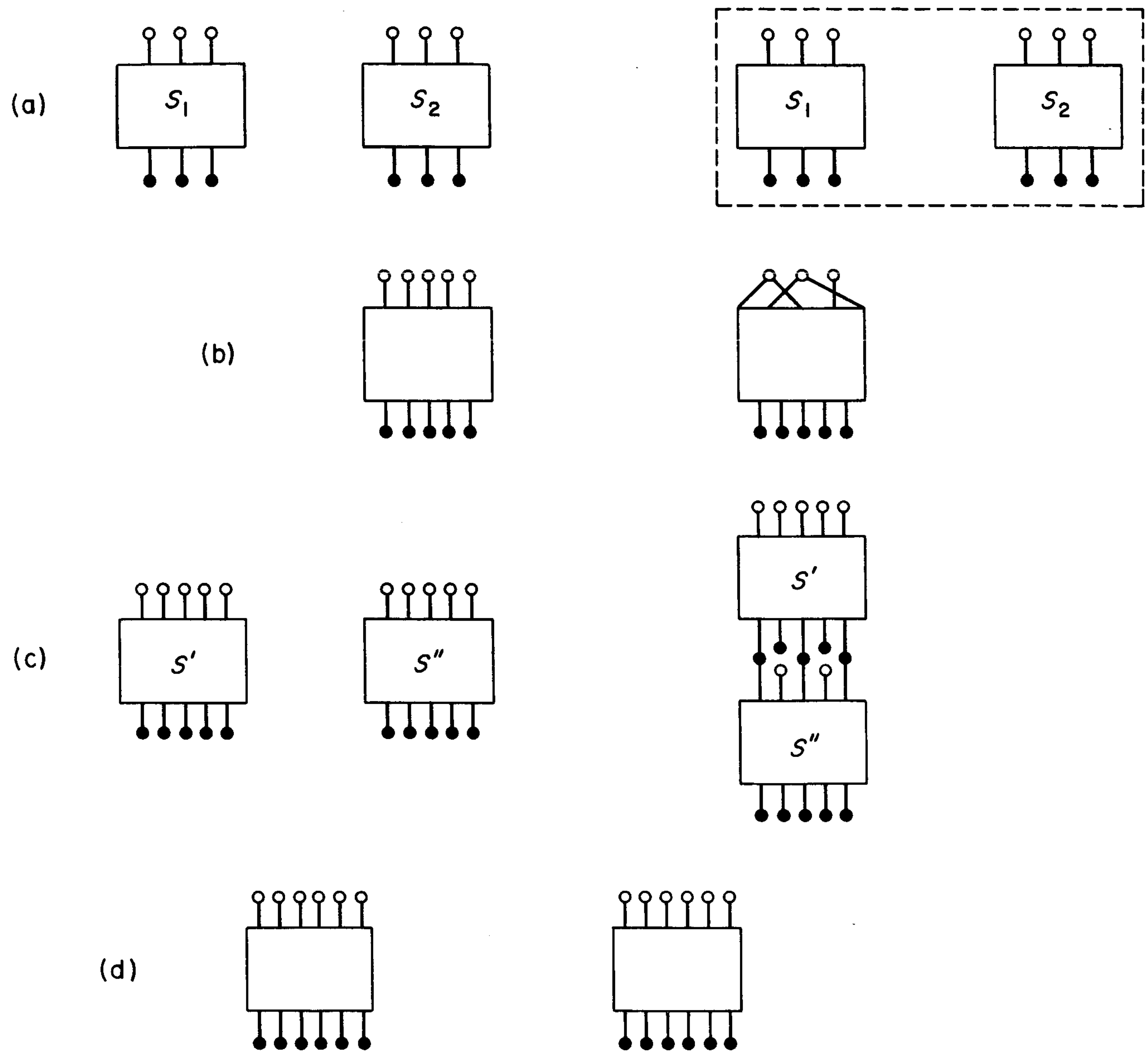


Fig. 7

essentially corresponds to the collection of vertices and terminals of the network S, if one does not distinguish vertices identified in the formation of the network S' (Fig. 7b).

4. If S' and S'' are networks without common vertices, with the corresponding collections of input and output terminals being M_1', M_2' and M_1'', M_2'', the result of pairwise identification of some collection N_2' (distinct in pairs) of output terminals of the network S' with some collection N_1'' (including as many elements again as in N_2') of similarly pairwise-distinct

input terminals of the network S'' will also be a network (S). The collection of its input terminals will be the set $M_1' \cup (M_1'' \setminus N_1'')$; the collection of its output terminals will be $M_2' \cup M_2''$. The collection of vertices of the network S coincides with the collection of vertices of the networks S' and S'' if one does not distinguish vertices identified in the formation of the network S (*cf.* Fig. 7c).

5. If S is a network, the result of designating as output terminals some subcollection of the collection of output terminals of the network S will also be a network (S'). The collection of vertices and input terminals of the network S' coincides with the corresponding collection of the network S (*cf.* Fig. 7d).

   We will say that the network S represents a network of the collection $\{E\}$ of functional elements, if all functional elements of the scheme S belong to the collection $\{E\}$.

Let a network have n input terminals $a_1, \cdots, a_n$. We associate with these the corresponding arguments $x_1, \cdots, x_n$. The function realized by the network is calculated by the following rules (we define the *designation* of a function from the set $(\sigma_1, \cdots, \sigma_n)$ of designations of the arguments):

1. To the input terminals of the network, $a_1, \cdots, a_n$, we ascribe the corresponding designations $\sigma_1, \cdots, \sigma_n$.

2. If the designations $t_1, \cdots, t_s$ are already ascribed (in order of the number of their input terminals), its output terminal is designated $f_E(t_1, \cdots, t_s)$, where f_E is the function realized by the functional element E.

From the definition of networks it follows that every vertex of the network will be ascribed some designation, and moreover uniquely. The designation ascribed to the vertex C appears by definition in that of the function $F_C(x_1, \cdots, x_n)$ from the set $(\sigma_1, \cdots, \sigma_n)$. By means of this function, we will exhibit that which corresponds to the output terminal of the network under consideration.

Let each functional element E be ascribed a non-negative real number $P(E)$ – its *weight.* The *index of simplicity* $L(S)$ of the network S is defined as the sum of the weights of all the functional elements in S.

The reduced weight $p(E)$ of the functional element E, for functions which essentially depend on $s \geqq 2$ arguments, will be described as the number $P(E)/(s-1)$.

The aim of the following paragraph is the description of a method of synthesis of a sufficiently economical (in the sense indicated by a higher index of simplicity) network for the realization of functions of algebraic logic. The collection $\{E\}$ of elementary functions, for which the networks are constructed, will be assumed to be *complete; i.e.*, it will be assumed that networks from $\{E\}$ can realize arbitrary functions of algebraic logic.

Let $L(n)$ be the set of numbers such that networks with index of simplicity not exceeding $L(n)$ can realize arbitrary functions of algebraic logic with n arguments.

Theorem 4. If $\{E\}$ is finite with all $P(E)$ positive, then*9

$$L(n) \sim (\min p(E)) \cdot \frac{2^n}{n},$$

there being for arbitrary $\epsilon > 0$ and $n > n(\epsilon)$ an allotted function f of the n arguments $x_1, \cdots, x_n$, for which $L(f) < (1 - \epsilon) L(n)$, tending to zero with increasing n.

The proof of the theorem breaks down into three parts. In § 5, an auxiliary assertion will be proved, utilized in the description of the method of synthesis. In § 6, the method of synthesis is described, and a bound from above is obtained. In § 7, the lower bound is proved.

*9 We recall that the function $p(E)$ is defined only for functions realizable from functional elements, essentially depending on more than one variable.

5. LEMMA

We introduce some notation. Establish, between sets $(\sigma_1, \cdots, \sigma_u)$ of zeros and ones of length u, and the whole numbers q, $0 \leq q < 2^u$, determined subsequently, the (reciprocally single-valued) correspondence: The number q corresponds to the set which exhibits the binary digits of the number q; that is,

$$q = \sum_{i=1}^{u} \sigma_i 2^{n-i} .$$

The number q corresponding to the set $(\sigma_1, \cdots, \sigma_u)$ will be designated $q(\sigma_1, \sigma_2, \cdots, \sigma_u)$, while the set corresponding to the number q will be designated $(\sigma_1(q), \cdots, \sigma_n(q))$.

Lemma. Let the function $F(x_0, \cdots, x_{N-1})$, $N \geq 2^u$, essentially depend on all N arguments. Then, the functions $\psi_i(y_1, \cdots, y_u, z)$, $0 \leq i < 2^u$, and (in the case that $N > 2^u$) $\chi_i(y_1, \cdots, y_u)$, $2^u \leq i < N$, also exist such that an arbitrary function of algebraic logic $f(y_1, \cdots, y_u, z_1, \cdots, z_k)$ can be represented in the form[*10]

$$\begin{gathered} f(y_1, \cdots, y_u, z_1, \cdots, z_k) \\ = F(\psi_0(y_1, \cdots, y_n, f(0, \cdots, 0, z_1, \cdots, z_k)), \cdots, \psi_{2^u-1}(y_1, \cdots, y_u, f(1, \cdots, 1, z_1, \cdots, z_k)), \\ \chi_{2^u}(y_1, \cdots, y_u), \cdots, \chi_{N-1}(y_1, \cdots, y_u)). \end{gathered} \tag{5.1}$$

Proof. Since the function F depends essentially on all its arguments, then for every j, $0 \leq j < 2^u$, there exist constants $C_{j,i}$, $0 \leq i < N$, such that[*11]

$$F(C_{j,0}, \cdots, C_{j,j-1}, x_j, C_{j,j+1}, \cdots, C_{j,N-1}) = x_j \oplus C_{j,j} . \tag{5.2}$$

We examine the functions ψ_i and χ_i, defined in the following manner:

$$\psi_i(\sigma_1(j), \cdots, \sigma_u(j), z) = \begin{cases} C_{j,i} & \text{if } j \neq i , \\ Z \oplus C_{j,i} & \text{if } j = i , \end{cases} \qquad (0 \leq i < 2^u), \tag{5.3}$$

[*10] The formula (5.1) represents its own generalization to the decomposition of functions on the arguments $y_1, \cdots, y_u$.

[*11] Here the sign $\oplus$ denotes addition modulo 2.

$$\chi_i(\sigma_1(j), \cdots, \sigma_u(j)) = C_{j,i}\,(2^u \leq i < N). \tag{5.4}$$

These functions satisfy the stated lemma. In the same way, we examine the arbitrary set $(\sigma_1, \cdots, \sigma_n)$. It exhibits the binary digits of the number $j = q(\sigma_1, \cdots, \sigma_m)$, $0 \leq j < 2^u$. From (5.2), (5.3), and (5.4) we have

$$F(\psi_0(\sigma_1(j), \cdots, \sigma_u(j), f(0, \cdots, 0, z_1, \cdots, z_k)), \cdots,$$

$$\psi_{2^u-1}(\sigma_1(j), \cdots, \sigma_u(j), f(1, \cdots, 1, z_1, \cdots, z_k)),$$

$$\chi_{2^u}(\sigma_1(j), \cdots, \sigma_u(j)), \cdots, \chi_{N-1}(\sigma_1(j), \cdots, \sigma_u(j)))$$

$$= F(C_{j,0}, \cdots, C_{j,j-1}, f(\sigma_1(j), \cdots, \sigma_u(j), z_1, \cdots, z_k) \oplus C_{j,j}, C_{i,j+1}, \cdots, C_{j,jN-1}) =$$

$$= f(\sigma_1(j), \cdots, \sigma_u(j), z_1, \cdots, z_k).$$

The lemma is proved.

Remark. It is evident that each of the functions ψ_i can be represented in the form

$$\psi_i(y_1, \cdots, y_u, z) = \bar{Z}\psi_{i0}(y_1, \cdots, y_u) \vee Z\psi_{i1}(y_1, \cdots, y_u), \tag{5.5}$$

where ψ_{i0} and ψ_{i1} are some functions of the arguments $y_1, \cdots, y_u$.

6. THE METHOD OF SYNTHESIS AND THE UPPER BOUND

Let us consider an arbitrary function of algebraic logic f of n arguments. It can be represented in the form

$$f = \bigvee_{j=1}^{p} f_j = \bigvee_{j=1}^{p} \bigvee_{\sigma_1, \cdots, \sigma_t} x_1^{\sigma_1} \cdots x_t^{\sigma_t} f_j(\sigma_1, \cdots, \sigma_t, x_{t+1}, \cdots, x_n), \tag{6.1}$$

where the functions f_j are relative to the K arguments $x_{n-k+1}, \cdots, x_n$ (*cf.* § 1); in the *proper representation* $f = \bigvee_{j=1}^{p} f_j$, it has the parameters k and s.

Let the elementary subnetwork E_0 with minimal reduced weight $p(E_0)$ realize the function ϕ, and let ϕ essentially depend on $r + 1 \geq 2$ arguments. We examine the function

$$F(y_0, \cdots, y_{vr}) = \phi(y_0, \cdots, y_{r-1}, \phi(y_r, \cdots \cdots \phi(y_{(v-1)r}, \cdots, y_{vr}) \cdots)),$$

where v is the minimum number such that $2^u - 1 \leq vr < 2^u - 1 + r$, while u satisfies the condition $u \leq n - t$.*12 The function F clearly depends essentially on all its arguments.

On the basis of the lemma of § 5, each of the functions $f_j(\sigma_1, \cdots, \sigma_t, x_{t+1}, \cdots, x_n)$ can be represented in the form

$$f_j(\sigma_1, \cdots, \sigma_t, x_{t+1}, \cdots, x_n) =$$

$$F(\psi_0(x_{+1}, \cdots, x_{t+u}, f_j(\sigma_1, \cdots, \sigma_t, 0, \cdots, 0, x_{t+u+1}, \cdots, x_n)), \cdots$$

$$\cdots, \psi_{2^u-1}(x_{t+1}, \cdots, x_{t+u}, f_j(\sigma_1, \cdots, \sigma_t, 1, \cdots, 1, x_{t+u+1}, \cdots, x_n)),$$

$$\chi_{2^u}(x_{t+1}, \cdots, x_{t+u}), \cdots, \chi_{vr}(x_{t+1}, \cdots, x_{t+u})), \tag{6.3}$$

where the functions ψ_i and χ_i do not depend on the set $(\sigma_1, \cdots, \sigma_t)$.

---

*12 From here it follows that

$$v < \frac{2^u}{r} + 1 \tag{6.2}$$

As already indicated, we suppose that the collection $\{E\}$ of elementary functions is complete. Therefore, there exist networks T_1, T_2, T_3, realizing the corresponding functions $F_1(x) = \bar{x}$, $F_2(x, y) = xy$, $F_3(x, y) = x \vee y$; the first of these networks has one input and one output; each of the remaining two has two inputs and one output. Let the indices of simplicity of these networks equal, respectively, l_1, l_2, l_3. We will utilize these networks in the process of synthesis in the capacity of "standard blocks." We will also employ the "standard block" T_4, realizing the function $F(y_0, \cdots, y_{vr})$ (with index of simplicity $l_4 = vP(E_0)$).*13

The network S for the function f we will construct from specific large blocks. Each of these will be constructed from standard blocks. The *lower reduction* describes these large blocks. The inputs of each of them either join to the previously described blocks, or appear as inputs of the networks (in succeeding cases it will be decreed that the arguments are placed in correspondence with them).

DESCRIPTION OF THE NETWORKS (combination of blocks schematically portrayed in Fig. 8).

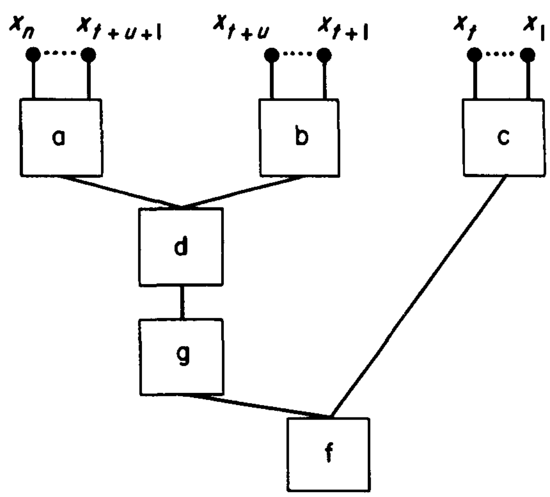


Fig. 8

*13 As will be seen from what follows, the networks will consist almost entirely of function elements with minimal reduced weight.

I. Block A realizes all functions $f_j(\sigma_1, \cdots, \sigma_{t+u}, x_{t+u+1}, \cdots, x_n)$. It has $k = n - t - u$ inputs, corresponding to the arguments $x_{t+u+1}, \cdots, x_n$ and constructed of the standard blocks T_1, T_2, T_3.

1) To begin with, the functions $\overline{x}_i (t + u + 1 \leq i \leq n)$ are realized; for them, k standard blocks T_1 are required. The index of simplicity of this part of the network equals $k l_1$.

2) Next, all conjunctions $x_{t+u+1}^{\sigma_{t+u+1}} \cdots, x_n^{\sigma_n}$ are realized. For the realization of each of them (proceeding from the functions x_i and $\overline{x}_i$), $k - 1$ standard blocks T_2 are required; for the realization of *all* the conjunctions, $(k-1)\,2^k$ standard blocks T_2 are required. The index of simplicity of this part of the network equals $(k-1)\,2^k\,l_2$.

3) Finally, we realize all functions $f_j(\sigma_1, \cdots, \sigma_{t+u}, x_{t+u+1}, \cdots, x_n)$ (not more than $p\,2^s \leq \left(\frac{2^k}{s} + 1\right) 2^s$ such). Every function, except for "identically zero," is realized as the disjunction of appropriate conjunctions; for them, not more than s standard blocks T_3 are required. "Identically zero" can be realized, for example, as $x_n \cdot \overline{x}_n$; for it, one standard block T_2 is needed. The index of simplicity for this part of the network does not exceed $s\left(\frac{2^k}{s} + 1\right) 2^s l_3 + l_2$.

In this manner, the index of simplicity $L(\mathrm{A})$ of Block A satisfies the relation

$$L(\mathrm{A}) \leq k l_1 + k 2^k l_2 + s\left(\frac{2^k}{s} + 1\right) 2^s l_3. \tag{6.4}$$

II. Block B realizes all conjunctions $x_{t+1}^{\sigma_{t+1}} \cdots x_{t+u}^{\sigma_{t+u}}$. It has u inputs, corresponding to the arguments $x_{t+1}, \cdots, x_{t+u}$, and 2^u outputs, and is moreover arranged as is the corresponding part of Block A. Its index of simplicity $L(\mathrm{B})$ evidently satisfies the relation

$$L(\mathrm{B}) \leq u l_1 + u\, 2^u l_2. \tag{6.5}$$

III. Block C realizes all conjunctions $x_1^{\sigma_1} \cdots x_t^{\sigma_t}$. It has t inputs corresponding to the arguments $x_1, \cdots, x_t$, and 2^t outputs, and is moreover arranged as is Block B. Its index of simplicity $L(\mathrm{C})$ satisfies the relation

$$L(\mathrm{C}) \leq t l_1 + t 2^t l_2. \tag{6.6}$$

IV. Block D realizes all functions $\psi_i(x_{t+1}, \cdots, x_{t+u}, f_j(\sigma_1, \cdots, \sigma_{t+u}, x_{t+u+1}, \cdots, x_n))$ and $\chi_i(x_{t+1}, \cdots, x_{t+u})$. The inputs to these blocks are the outputs of Block B and some of the outputs of Block A.

1) For the realization of every function χ_i (as a disjunction of suitable conjunctions) not more than 2^u standard blocks T_3 are required. The number of functions χ_i does not exceed r (*cf.* the previous page). Therefore the index of simplicity of this part of Block D is not more than $r\, 2^u\, l_3$.

2) We have (*cf.* the remark after the lemma of § 5)

$$\psi_i(x_{+1}, \cdots, x_{t+u}, f_j(\sigma_1, \cdots, \sigma_{t+u}, x_{t+u+1}, \cdots, x_n))$$

$$= f_j(\sigma_1, \cdots, \sigma_{t+u}, x_{t+u+1}, \cdots, x_n) \cdot \psi_{i1}(x_{t+1}, \cdots, x_{t+u})$$

$$\bigvee f_j(\sigma_1, \cdots, \sigma_{t+u}, x_{t+u+1}, \cdots, x_n) \cdot \psi_{i0}(x_{t+1}, \cdots, x_{t+u})$$

For the realization of each of the functions ψ_{i0} and ψ_{i1} $\left(\text{formed from the conjunctions } x_{t+1}^{\sigma_{t+1}} \cdots x_{t+u}^{\sigma_{t+u}}\right)$ not more than 2^u standard blocks T_3 are required; for the realization of each of the functions $\psi_i(x_{t+1}, \cdots, x_{t+u}, f_j(\sigma_1, \cdots, \sigma_{t+u}, x_{t+u+1}, \cdots, x_n))$ (formed from ψ_{i0} and ψ_{i1} and the functions realized by Block A), two standard blocks T_2 and one each of the standard blocks T_1 and T_3. The number of functions ψ_i equals 2^u; the number of distinct functions $f_j(\sigma_1, \cdots, \sigma_{k+t}, x_{k+t+1}, \cdots, x_n)$ does not exceed $\left(\frac{2^k}{s} + 1\right) 2^s$. Therefore the index of simplicity for the part of Block D realizing all functions $\psi_i(x_{k+1}, \cdots, x_{k+t}, f_j(\sigma_1, \cdots, \sigma_{k+t}, x_{k+t+1}, \cdots, x_n))$, does not exceed

$$2^u \left(\frac{2^k}{s} + 1\right) \times 2^s \; [l_1 + 2l_2 + (2^{u+1} + 1)\, l_3].$$

The index of simplicity L(D) of Block D satisfies the relation

$$L(\mathrm{D}) \le r\, 2^u\, l_3 + 2^u \left(\frac{2^k}{s} + 1\right) 2^s \; [l_1 + 2l_2 + (2^{u+1} + 1)\, l_3]. \tag{6.7}$$

V. Block G realizes all the functions $f_j(\sigma_1, \cdots, \sigma_t, x_{t+1}, \cdots, x_n)$, formed from the functions realized by Block D. For the realization of each of them, one standard block T_4 is required [*cf.* (6.3)]. Therefore

$$L(G) \leq 2^t \left(\frac{2^k}{s} + 1\right) l_4 . \tag{6.8}$$

VI. Block F realizes the function f, formed from the functions realized by Blocks C and G [*cf.* (6.1)]. It contains not more than $2^t \quad \frac{2^k}{s} + 1$ standard blocks T_2 [for the realization of each conjunction $x_1^{\sigma_1} \cdots x_t^{\sigma_t} f_j(\sigma_1, \cdots, \sigma_t, x_{t+1}, \cdots, x_n)$] and not more than $2^t \left(\frac{2^k}{s} + 1\right)$ standard blocks T_3 (for the realization of these conjunctions of the function f). Therefore

$$L(F) \leq 2^t \left(\frac{2^k}{s} + 1\right) (l_2 + l_3). \tag{6.9}$$

From (6.4), (6.5), (6.6), (6.7), and (6.9) we have

$$L' = L(A) + L(B) + L(C) + L(D) + L(F) \leq N_1 l_1 + N_2 l_2 + N_3 l_3, \tag{6.10}$$

where

$$N_1 = k + u + t + 2^{u+s} \left(\frac{2^k}{s} + 1\right), \tag{6.11}$$

$$N_2 = k2^k + u2^u + t2^t + 2 \cdot 2^{u+s} \left(\frac{2^k}{s} + 1\right) + 2^t \left(\frac{2^k}{s} + 1\right), \tag{6.12}$$

$$N_3 = s2^s \left(\frac{2^k}{s} + 1\right) + r\,2^u + 2^{u+s} \left(\frac{2^k}{s} + 1\right) \left(2^{u+1} + 1\right) + 2^t \left(\frac{2^k}{s} + 1\right). \tag{6.13}$$

We now set

$$k = [2 \log_2 n], \quad u = [\log_2 n], \quad s = [n - 5 \log_2 n]; \tag{6.14}$$

then

$$t = n - k - u < n - 3 \log_2 n + 2 . \tag{6.15}$$

We introduce the notation

$$\frac{n^2}{n - 5\log_2 n - 1} + 1 = N_0 \,. \tag{6.16}$$

From (6.11), (6.12), (6.13), (6.14), (6.15) and (6.16) there follows:

$$N_1 \leq n + \frac{2^n}{n^4} N_0 = O\left(\frac{2^n}{n^2}\right) , \tag{6.17}$$

$$N_2 \leq 2n^2 \log_2 n + n \log_2 n + (n - 3\log_2 n + 2) \frac{4.2^n}{n^3} + \frac{2.2^n}{n^4} N_0 + \frac{4.2^n}{n^3} N_0 = O\left(\frac{2^n}{n^2}\right) , \tag{6.18}$$

$$N_3 \leq (n - 5\log_2 n) \frac{2^n}{n^5} N_0 + r\,n + \frac{2^n}{n^4}(2n+1) N_0 + \frac{4.2^n}{n^2} N_0 = O\left(\frac{2^n}{n^2}\right) . \tag{6.19}$$

From (6.10), (6.17), (6.18), and (6.19) it follows that

$$L' = O\left(\frac{2^n}{n^2}\right) . \tag{6.20}$$

From (6.8) and (6.2) we have:

$$L(\mathrm{G}) \leq 2^t \left(\frac{2^k}{s} + 1\right)\left(\frac{2^u}{r} + 1\right) P(E_0) = \frac{2^{t+k+u}}{rs}(1 + o(1))\, P(E_0) = P(E_0)\frac{2^n}{n}(1 + o(1)), \tag{6.21}$$

because $t + k + u = n$ and $\dfrac{P(E_0)}{r} = p(E_0)$. It is clear that

$$L(\mathrm{S}) = L' + L(\mathrm{G}) .$$

Therefore [*cf.* (6.20) and (6.21)]

$$L(\mathrm{S}) \leq p(E_0) \frac{2^n}{n} (1 + o(1)) .$$

7. THE LOWER BOUND

In the work of the author [4], a theorem is proved concerning lower bounds for the index of simplicity of networks of arbitrary elements. However, in our present case we get bounds from it which are insufficient for the proof of the theorem. Therefore, another proof is given here.

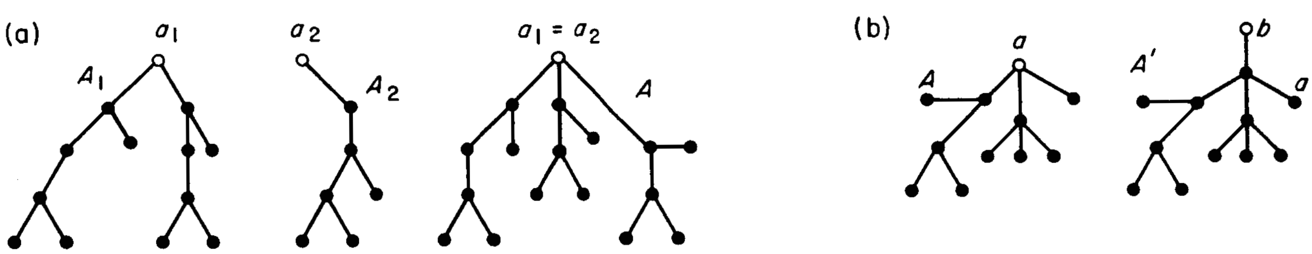
We calculate the number of functions of n arguments, $x_1, \cdots, x_n$, realized by networks with index of simplicity not exceeding L.

1. We will consider trees with roots [17], *i.e.* trees [15] at each of whose outputs there is one vertex which is the *root.*

Lemma 1. The number $S(h)$ of trees with roots, having h ribs, satisfies the relation[*14]

$$S(h) \leq 4^n$$

Proof. The collection of trees with roots can inductively be determined in the following manner: a) a rib with one marked vertex is a tree with a root; b) if A_1 and A_2 are trees (without common vertices), with corresponding roots a_1 and a_2, the result A of identifying the vertices a_1 and a_2 is a tree with root $a_1 = a_2$ (*cf.* Fig. 9a); c) if A is a tree with root a, the result A' of annexing to a the rib ab (where b is not a vertex of A) is a tree with root b (*cf.* Fig. 9b).



[*14] A more precise bound is obtained in [17] with the help of generating functions.

To each of the trees with roots can be juxtaposed (generally speaking, not uniquely) a word from the alphabet $\{\alpha, \beta\}$ in the following manner (in accordance with the inductive structure of these trees): a') to the rib of step a), juxtapose the word $\alpha\beta$; b') if to the trees A_1 and A_2 of step b) are juxtaposed the corresponding words A_1 and A_2, then to the tree A is juxtaposed the word $\mathrm{A}_1\mathrm{A}_2$; c') if to the tree A of step c) is juxtaposed the word A, then to the tree A' corresponds the word $\alpha \mathrm{A} \beta$.

For example, to the tree of Fig. 10 can be juxtaposed the word

$$\alpha\alpha\underbrace{\alpha\alpha\beta\alpha\beta\alpha\beta\alpha\beta\beta\alpha}_{\mathrm{A}_1}\beta\beta\alpha\beta\alpha\beta\beta\underbrace{\alpha\alpha\alpha\beta\alpha\beta\alpha\beta\beta\alpha\beta\beta\alpha\beta}_{\mathrm{A}_2} \cdot$$

In it appear the subwords A_1 and A_2, corresponding to the subtrees A_1 and A_2.

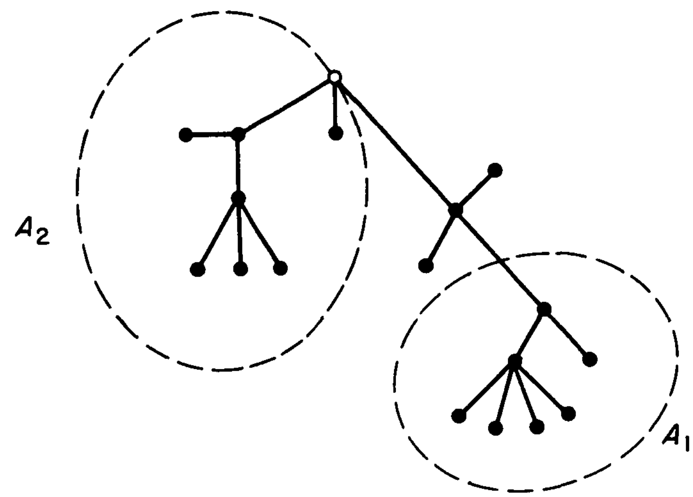


Fig. 10

We note that on the word A, corresponding to the tree A with root, the latter is reestablished uniquely (with exactness up to isomorphism [15], root is transformed into root).

To a tree with h ribs corresponds a word of length 2^h. Therefore, the number of trees with roots, having h ribs, does not exceed the number of words of length $2h$ from the alphabet $\{\alpha, \beta\}$, *i.e.* $2^{2h} = 4^h$. The lemma is proved.

Corollary. The number $S(h, n)$ regulating (*i.e.* enumerating) systems of n trees with roots (some trees can be vacuous) having in aggregate h ribs, satisfies the relation

$$S(h, n) \leq (h + n)^n \, 4^h \ .$$

In fact, the number of ways of decomposing the number h into n integral nonnegative parts $h^{(1)}, \cdots,$ $h^{(n)}$ $\left(\sum_{i=1}^{n} h^{(i)} = h\right)$ equals the number of combinations with repetition of $h+1$ elements by $n=1$,[*15] *i.e.* $C_{n+h-1}^{n-1} < (n+h)^n$; every such decomposition corresponds to $S(h^{(1)}) \cdot S(h^{(2)}) \cdot \cdots \cdot S(h^{(n)}) \leq 4^h$ systems of trees (here $S(0) = 1$).

2. We will now consider oriented graphs [15], not containing oriented cycles (*i.e.* subgraphs consisting of edges $(a_{i_1}\ a_{i_2}), (a_{i_2}\ a_{i_3}), \cdots, (a_{i_{k-1}}\ a_{i_k}), (a_{i_k}\ a_{i_1})$). Vertices with each of their incident edges oriented in the direction *toward* them (inputs in their edges) will be called the *introverted vertices* of the graph; the remaining vertices are *inputs.* The *order* of an introverted vertex is what we call the number of incidental edges which are directed toward it.

Let the graph G have h_s introverted vertices of order s, $1 \leq s \leq m$, where m is the maximum of their orders. The set $H = (h_1, h_2, \cdots, h_m)$ will be called the *order structure* of the graph, and the number $\mu(H) = \Sigma\, sh_s / \Sigma\, h_s$ is the *average order* of the graph (in other words, the average order is the average arithmetical order of the introverted vertices).

We denote by $R(n, H)$ the number of nonisomorphic [15] oriented graphs without oriented cycles with n distinguishable (*i.e.* numbered) input vertices, and order structure H.

Lemma 2.

$$R(n, H) < 8^{\mu(H)h}\ (h+n)^{(\mu(H)-1)h+n}$$

where

$$h = \Sigma\, h_s$$

Proof. In every star of an introverted vertex of an arbitrary oriented graph without oriented cycles, we distinguish a single input among its edges. The number of these edges equals h. From the collection an ordered system of n trees with roots is formed. Therefore *any* oriented graph with order structure H with n distinguishable input vertices can be obtained from some ordered system of n trees with roots in the following manner. We designate the vertices of this system as roots of ascribed order: the order s is ascribed to h_s

[*15] That is, the number of ways of arranging $n-1$ "commas" before, after, and between the digits of an h-digit number; the number of digits between successive "commas" equals the corresponding part; between successive digits several "commas" may be inserted — for example the decomposition $7 = 0 + 0 + 1 + 3 + 2 + 0 + 1 + 0$ corresponds to the arrangement of "commas",, 1, 111, 11,,1, . These arrangements of "commas" are combinations of "empty spaces" before, after, and between the digits of an h-digit number with repetition.

vertices (the number of ways of assigning the orders equals the number of ways of decomposing the number $h' = \mu(H)h = \sum_{s=1}^{m} sh_s$ into corresponding numbers of ordered natural terms, each of which does not exceed m, *i.e.* this does not exceed $2h'$).*[16] Next, every vertex of order s connects $s-1$ edges directed to it with some other vertices; the number of ways of connecting each vertex does not exceed the number of combinations with repetition from $h+n$ elements by $s-1$, *i.e.* $C_{h+s+n-2}^{s-1}$; the number of ways of connecting all introvert vertices does not exceed*[17]

$$\prod_{s=1}^{m} (C_{h+n+s-2}^{s-1})^{h_s} \leq \prod_{s=1}^{m} (h+n)^{(s-1)h_s} = (h+n)^{h'-h} .$$

In general, the number of graphs with order structure H with n input vertices, therefore, does not exceed (lemma 1)

$$(h+n)^n \cdot 4^h \cdot 2^{h'} (h+n)^{h'-h} \leq 8^{h'} (h+n)^{h'-h+n} .$$

The lemma is proved.

3. We replace, in the network of functional elements, each of the last stars of edges, oriented toward the center of the star, in number equal to the number of input functional elements (the center of the star corresponds to the output terminal of the functional elements, the remaining vertices, to the input terminals); after that we write under the center of each star a symbol, denoting the functional element corresponding to it, and its edge number, corresponding to the number of representations by them of inputs of functional elements (*cf.* Fig. 11). The oriented graph obtained by virtue of the definition of networks from functional elements will not be kept in cyclic orientation. Obviously, between ordered graphs (with "numbered through" vertices and edges) and elementary networks, singlevaluedness is restored.

The index of simplicity of a graph is defined as the sum of the weights of the functional elements whose symbols are written down under its vertices (*i.e.*, its index of simplicity equals the index of simplicity of the resulting network).

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*[16] It does not exceed the number of ways of inserting "commas" among the digits of an h'-digit number (*cf.* the preceding footnote), in such a way that between successive digits not more than one comma is inserted.

*[17] For $h \leq 1$, n must always be not less than 1.

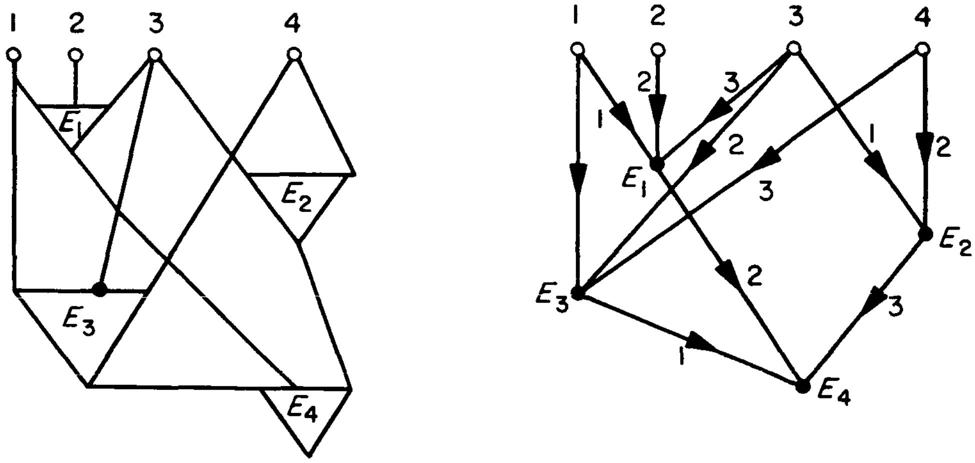

Fig. 11

4. We introduce some notation.

$Q(n, L)$ is the number of functions of algebraic logic in n arguments $x_1, \cdots, x_n$ realized by networks of functional elements from $\{E\}$ with index of simplicity not exceeding L. $Q'(n, L)$ is the number of networks made of functional elements from $\{E\}$ with index of simplicity which does not exceed L, realizing functions of algebraic logic in n arguments $x_1, \cdots, x_n$. $Q''(n, L)$ is the number of oriented graphs with n distinct input vertices, not containing oriented cycles, and with introverted vertices under which are written symbols for the functional elements of $\{E\}$, and edges under which are written integers from 1 to m, having an index of simplicity not exceeding L.

From what was said above, it follows that

$$Q(n, L) \leq Q'(n, L) \leq Q''(n, L) \tag{7.1}$$

5. Let $\{E\}$ consist of functional elements E_{ij}, $1 \leq j \leq M_i$, $1 \leq i \leq m$; the element E_{ij} has i input terminals and its weight equals P_{ij}.

We designate by h_{ij} the number of elements, E_{ij} in the graph G, and set $h = \Sigma\, h_{ij}$. We obtain an upper bound for some auxiliary function of order structure for a graph, having index of simplicity $\left(\text{equal to } \sum_{i=1}^{m} \sum_{j=1}^{M_j} h_{ij} P_{ij}\right)$, not exceeding L. We define*[18] $\rho = \min\limits_{i \geq 2; j} \dfrac{P_{ij}}{i-1}$. Thus for arbitrary i and j ($2 \leq i \leq m$, $1 \leq j \leq M_i$) $i - 1 \leq \dfrac{1}{\rho} P_{ij}$. Hence

*[18] We recall that all P_i are positive.

$$(\mu(h)-1)\,h = \sum_{i=2}^{m} (i-1) \sum_{j=1}^{M_i} h_{ij} \le \sum_{i=2}^{m} \sum_{j=1}^{M_i} \frac{1}{\rho} h_{ij} P_{ij} \le \frac{1}{\rho} L \,. \qquad (7.2)$$

Moreover, if $\Pi = \min_{i,j} P_{ij}$, then

$$h = \sum_{i=1}^{m} \sum_{j=1}^{M_i} h_{ij} \le \frac{1}{\pi} \sum_{i=1}^{m} \sum_{j=1}^{M_j} h_{ij} P_{ij} = \frac{1}{\pi} L \,. \qquad (7.3)$$

6. We will compute the number $H(L)$ of order structure H for graphs with index of simplicity not exceeding L. This number does not exceed the number of nonnegative integers satisfying the inequality

$$\sum_{i=1}^{m} \sum_{j=1}^{M_i} h_{ij} P_{ij} \le L \,,$$

which does not exceed the number satisfying the inequality

$$\sum_{i=1}^{m} \sum_{j=1}^{M_i} h_{ij} \le \frac{1}{\pi} L \,,$$

because $P_{ij} \ge \pi$. The number satisfying the last inequality does not exceed the number of ways of decomposing the number $\left[\frac{1}{\pi} L\right]$ into $M+1$ ordered nonnegative integral parts, where $M = \sum_{i=1}^{m} M_i$, *i.e.*[*19]

$$C^{M}_{\left[\frac{1}{\pi} L\right]+M} \le \left(\frac{1}{\pi} L + M\right)^{M}.$$

Thus,

$$H(L) \le \left(\frac{1}{\pi} L + M\right)^{M}. \qquad (7.4)$$

7. It is clear that

$$Q''(n, L) \le H(L) \cdot R(n, H) \cdot M^{h} m^{h} \,. \qquad (7.5)$$

[*19] *Cf.* footnote 15.

From lemma 2, (7.2), (7.3), (7.4) and (7.5) it follows that

$$Q''(n, L) \leq \left(\frac{1}{\pi} L + M\right)^{M} \cdot 8^{\mu(H)h} (h+n)^{(\mu(H)-1)h+n} (Mm)^{h}$$

$$\leq \left(\frac{1}{\pi} L + M\right)^{M} 8^{\left(\frac{1}{\pi}+\frac{1}{\rho}\right) L} \left(\frac{1}{\pi} L + n\right)^{\frac{1}{\pi} L+n} (Mm)^{\frac{1}{\pi} L} . \tag{7.6}$$

An immediate examination can convince, as a consequence of (7.6), that for $L \leq \rho \dfrac{2^n}{n} (1-\epsilon)$ [for arbitrary $\epsilon > 0$ and $n > n(\epsilon)$]

$$\frac{Q''(n, L)}{2^{2^n}} \to 0 \text{ as } n \to \infty .$$

Finally, taking (7.1) into account, we obtain a lower bound for $L(n)$, asymptotically equal to the upper one produced in § 6 (because ρ is the minimum reduced weight for functional elements), and the second part of Theorem 4 is proved.

Theorem 4 is completely proved.

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(NOTE: DAN = Doklady Akadamii Nauk)